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M-integral analysis for two-dimensional solids with strongly interacting microcracks. Part I: in an infinite brittle solid

Yi-Heng Chen *

Department of Engineering Mechanics, School of Civil Engineering and Mechanics, Xi'an Jiaotong University, Xi'an, Shaanxi 710049, People's Republic of China

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Abstract

This paper addresses an alternative description for brittle solids with strongly interacting microcracks. The basic idea starts from the M-integral analysis customarily used in single crack problems. As an initial attempt, the discussion is limited to the infinite two-dimensional cases and the microcracks are assumed to be stationary. It is proved from the global–local coordinate translations that the M integral is divided into two distinct parts. First of them is induced from the well-known relation between the integral and the stress intensity factors (SIFs) at all the crack tips (Freund, 1978). The second is contributed from the two components of the J_k vector (Knowles and Sternberg, 1972, Budiansky and Rice, 1973) and the coordinates of each microcrack center. The later is concerned not only with the crack tip SIFs, but also with the contribution arising from the traction-free surfaces of each crack (Herrmann and Herrmann, 1981). A detailed proof for the vanishing nature of the J_k vector along a closed contour surrounding all the microcracks is presented, from which the confusion about the dependence of the M integral on the origin selection of global coordinates is clarified. Two numerical examples are shown in tables and figures to confirm the derived conclusions. It is shown that the M integral is equivalent to the decrease of the total potential energy of the microcracking solids although the strongly interacting situations are taken into account. Therefore, a simple relation between the M integral and the L integral is established under the assumption mentioned above. It is concluded that the M-integral analysis, from the physical point of view, does play important role and provide an effective measure in evaluating the damage level of brittle solids with strongly interacting and randomly distributed microcracks. Although only the stationary microcracks are considered in the present investigation, the derived conclusions could actually be extended to treat much more useful problems, in which the multi-cracks may become critical and may grow during loading histories.

Keywords: M integral; J_k vector; Conservation law; Microcracks

* Tel.: +86-29-2660404; fax: +86-29-323-7910.
E-mail address: yhchen2@xjtu.edu.cn (Y.-H. Chen).

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1. Introduction

Nonlinear mechanical responses of microcracking solids have received considerable attention in the past 40 years (e.g., Kachanov, 1958; Kachanov, 1992, 1993; Chaboche, 1988a,b; Krajcinovic, 1989; Gross, 1982; Chen, 1984; Horii and Nemat-Nasser, 1983,1985; Mori and Tanaka, 1973; Nemat-Nasser and Hori, 1990; Benveniste, 1986; Christensen, 1990; Chen and Hasebe, 1998; Jun, 1991; Jun and Lee, 1991; Jun and Chen, 1994a,b; Budiansky and O'connell, 1976). As is now generally accepted, the existence, the growth and the nucleation of microcracks in brittle solids are of significant importance in both mechanical and civil engineering. The earliest work on this subject was given by Kachanov (1958) and several literature reviews were given by Chaboche (1988a,b), Krajcinovic (1989), and Kachanov (1992, 1993). Now, it is the commonly recognized opinion that microcracks in brittle materials, e.g., concrete, rocks and ceramics, often control overall deformation and failure mechanism, because the distributed microcracks in such materials not only lead to macrocrack initiation and final fracture, but also induce progressive material damage (deterioration or evolutionary damage). Nevertheless, the progressive material deterioration could be measured through the decrease of strength, stiffness, toughness, stability and residual life of such materials. Chaboche (1988a,b) reviewed some general features to continuum damage mechanics and summarized its main possibilities to present definitions and measures of damage, the description of the mechanical behavior of evolution of the corresponding damage variables. Moreover, there have been a number of investigators, who focus their attention on the effective elastic moduli of two-dimensional (2D) brittle solids with strongly interacting, randomly distributed microcracks. For example, Jun and Chen (1994a,b) presented their statistical micro-mechanical formulations to investigate the effective elastic moduli of 2D brittle solids with interacting slit microcracks. Besides these, the nonlocal damage theory based on micromechanics of crack interactions was proposed (e.g., Bazant, 1986; Bazant and Cedolin, 1991; Zdenek and Bazant, 1992). The three-dimensional problems were also treated (Budiansky and O'connell, 1976; Jun and Lee, 1991; Kachanov and Laures, 1989; Kachanov, 1993).

On the other hand, within the framework of plane, linear fracture mechanics many path-independent integrals were proposed. Among them, Rice's J integral as well as the J_k vector, the L integral, and the M integral are extremely attractive (Rice, 1968; Knowles and Sternberg, 1972; Budiansky and Rice, 1973; Kanninen and Popelar, 1985). However, to the present author's knowledge, the above mentioned path-independent integrals were always limited in single crack problems and no one in the open literature had accounted for the roles the integrals play in multi-cracks interaction problems or microcrack damage problems. Particularly, the implicit relation between the effective elastic moduli and the above mentioned path-independent integrals remains unknown even in the simplest cases, where all microcracks are assumed to be stationary (in other words, no microcracks are allowed to grow or nucleated during loading histories, the so-called "stationary" micro-mechanical models).

The purpose of the present work is not to present some advances in the continuum damage mechanics, rather, the purpose is to supply the lack of investigation of conservation laws in microcrack damage problems. After doing so, an alternative description or a new evaluation for microcrack damage in brittle solids could be established. The basic idea starts from the M-integral analysis customarily used in single crack problems (Knowles and Sternberg, 1972; Budiansky and Rice, 1973). The closed contour chosen to calculate the integral either encloses all microcracks or encloses a typical microcrack completely. As an initial attempt, the simplest cases only concerned with the stationary microcracks are considered and all the microcracks are assumed to be fully open and not intersecting with each other. In Section 2, a detailed proof for the vanishing nature of the J_k vector is given under the assumption that the closed contour chosen to calculate the J_k vector encloses all the microcracks (or there are no other discontinuities outside of the closed contour). And then, under the same assumption some lengthy manipulations are performed to prove the independence of the M integral from the origin selection of global coordinates. It is found from the global–local coordinate translations that the M integral is divided into two distinct parts for a cloud of microcracks in a 2D brittle solid. First of them is corresponding to the contribution arising from SIFs of each crack formulated by Freund (1978), while the second is corresponding to the contribution induced from the two components of the J_k vector and the center coordinates of each crack in the global coordinates. In Section 3, two examples are presented. They are concerned with four regularly distributed microcracks and 20 randomly distributed microcracks. Numerical results are shown in tables and figures, which not only confirm the independence of the M integral from the selection of the global system, but also reveal the equivalence between the M integral and the decrease of the total potential energy of the solid even under strongly interacting crack situations. Therefore, a simple relation between the M integral and the L integral in the present problem is derived under the assumption mentioned above.

Of the most interest is the implicit relation between the M integral and the reduction in the effective elastic moduli. It is found that the value of the integral against the remote loading angle increases as the moduli decrease and vice versa. The loading direction at which the maximum value of the integral occurs coincides well with the largest loss in effective moduli. Particularly, numerical results for 20 randomly distributed microcracks show that the value of the M integral is not sensitive to the remote loading direction. This coincides well with the well-known conclusion that the randomly distributed microcracks produce an isotropic overall elastic response (Jun and Chen, 1994a).

Although the present investigation is an initial attempt, which has not been proved to be universal to include all aspects of microcracking brittle solids, it does provide an evidence and a possibility for establishing an alternative description or a new evaluation for microcracking damage based on the M-integral analysis. Indeed, the present work reveals that the M integral does play an important role in damage mechanics. At least, it provides an effective measure for the damage level of brittle solids with strongly interacting microcracks in the so-called "stationary" micro-mechanical models. It is concluded that an inherent relation does exist between the M integral and the effective elastic moduli for a microcracking brittle solid. In other words, the effective elastic properties could be reformulated by the M integral, although the detailed mathematical formulation between the moduli and the M integral remains to be adequately investigated.

2. Independence of the M integral from the origin selection of the global coordinates

The definition of the M integral is formulated as (Knowles and Sternberg, 1972; Budiansky and Rice, 1973):

$$M = \oint_C (wx_i n_i - T_l u_{l,i} x_i)\,\mathrm{d}s, \tag{1}$$

which is customarily adopted in single crack problems (Herrmann and Herrmann, 1981). Here, w is the strain energy density and T_l is the traction acting on the outside of a closed contour C, $u_{l,i} = \partial u_l/\partial x_i$ $(l = 1,2;\quad i = 1,2)$.

Consider a multi-cracks problem as shown in Fig. 1. Assume that the closed contour C encloses all the cracks completely in a 2D brittle solid. A global coordinate system, say (x_1, x_2), with the origin O, is introduced (Fig. 1). Here, it should be emphasized that Eq. (1) is defined in the global coordinate system (x_1, x_2). Furthermore, a typical microcrack (k) is considered in Fig. 1 and a local coordinate system $(x_1^{(k)}, x_2^{(k)})$ is then introduced, which is parallel to (x_1, x_2), but its origin $O(k)$ is different from the origin O. Here, $\xi_1^{(k)}$ and $\xi_2^{(k)}$ are coordinates of the origin $O(k)$ in the global system (x_1, x_2), and the following relations are valid:

$$\begin{aligned} x_1 &= \xi_1^{(k)} + x_1^{(k)}, \\ x_2 &= \xi_2^{(k)} + x_2^{(k)}. \end{aligned} \tag{2}$$

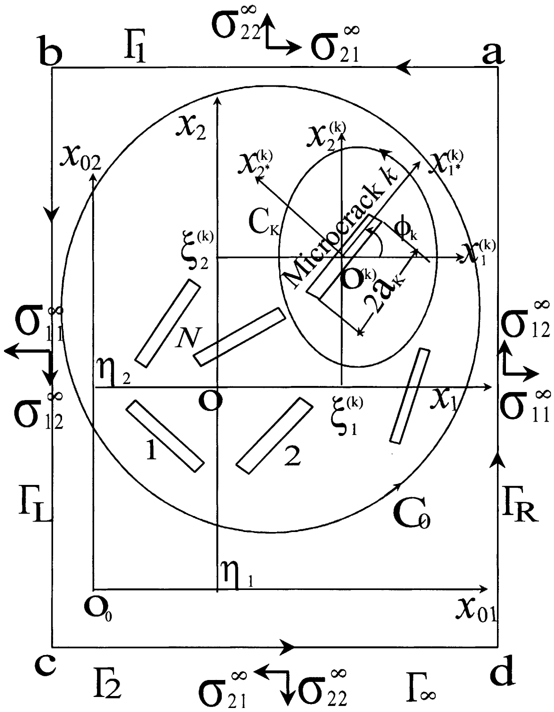

Fig. 1. Strongly interacting microcracks in a 2D-brittle solid and the closed contours specially introduced.

In order to define the M integral in the local system $(x_1^{(k)}, x_2^{(k)})$, a special closed contour $C^{(k)}$ is also introduced, which encloses the microcrack (k) only. Thus

$$M^{(k)} = \oint_{C^{(k)}} \left(w x_i^{(k)} n_i^{(k)} - T_l^{(k)} \cdot u_{l,i}^{(k)} \cdot x_i^{(k)} \right) \mathrm{d}s \quad (k = 1, 2, \ldots, N), \tag{3}$$

where the superscript (k) denotes the quantities defined in the local system $(x_1^{(k)}, x_2^{(k)})$.

Here, it has been assumed that all the microcracks are not intersecting with each other although the microcrack concentrations are higher and microcrack spaces are closer. As pointed out by Jun and Chen (1994a,b), in this case, the strong microcrack interactions occur and the effective medium theories are very cumbersome to be appropriated. The multi-crack interaction problem shown in Fig. 1 could easily be solved as treated by Gross (1982) and Chen (1984). The detailed numerical technique is no longer discussed here. After doing so, the stress intensity factors at both tips of the kth microcrack denoted by $K_{\mathrm{IR}}^{(k)}$, $K_{\mathrm{IIR}}^{(k)}$, $K_{\mathrm{IL}}^{(k)}$, $K_{\mathrm{IIL}}^{(k)}$ $(k = 1, 2, \ldots, N)$ could be given and the $M^{(k)}$ integral could then be evaluated by using Freund's formulation (1978):

$$M^{(k)} = \frac{\kappa + 1}{8\mu} \left[\left(K_{\mathrm{IR}}^{(k)} \right)^2 + \left(K_{\mathrm{IIR}}^{(k)} \right)^2 + \left(K_{\mathrm{IL}}^{(k)} \right)^2 + \left(K_{\mathrm{IIL}}^{(k)} \right)^2 \right] a_k, \tag{4}$$

where μ is the shear modulus of the brittle solid and $\kappa = 3 - 4v$ for plane strain, v is the Poisson's ratio, subscripts R and L denote the stress intensity factors at the right tip and left tip of the microcrack (k), respectively, subscripts I and II denote the Mode I and Mode II fracture, respectively, and a_k refers to the half length of the kth microcrack.

It is well known (Appendix A) that the value of $M^{(k)}$ is independent of the rotation of the local system, when it is transferred from $(x_1^{(k)}, x_2^{(k)})$ to $(x_{1^*}^{(k)}, x_{2^*}^{(k)})$ with an oriented angle φ_k (Fig. 1). However, it is also seen that the total contribution induced from the N microcracks to the M integral could not be given by using a simple summation among $M^{(k)}$ $(k = 1, 2, \ldots, N)$. This means that

$$M \neq \sum_{k=1}^{N} M^{(k)}, \tag{5}$$

which does lead to a significant trouble for evaluating the M integral in multi-crack problems.

Indeed, it is not clear whether the M integral depends on the origin selection of the global coordinate system as many previous researchers suspected. Moreover, how to calculate the value of the M integral for multi-cracks problems is worthy of further investigation. For further details, the J_k vector should be introduced here:

$$J_k = \oint_C (w n_k - u_{i,k} T_i)\, \mathrm{d}s \quad (k = 1, 2), \tag{6}$$

where $k = 1$, 2 denote the two components of the vector, i.e., $J_1 = J$ and J_2.

Although Herrmann and Herrmann (1981) found the path-dependent nature of J_2 for a closed contour surrounding only single tip of a finite crack, the forthcoming manipulations are always corresponding to such a case, in which the closed contours either enclose all the microcracks or enclose a typical microcrack completely. Therefore, both J_1 and J_2 are path independent although, as pointed out by Herrmann and Herrmann (1981) and as will be seen below, the traction-free surfaces of the microcracks have contributions to J_2.

The components of the vector expressed by Eq. (6) are dependent on the rotation of the coordinate system from $(x_1^{(k)}, x_2^{(k)})$ to $(x_{1^*}^{(k)}, x_{2^*}^{(k)})$ with an oriented angle φ_k (Fig. 1). The values of $J_k^{(k)}$ evaluated along $C^{(k)}$, respectively in the local systems $(x_1^{(k)}, x_2^{(k)})$ and $(x_{1^*}^{(k)}, x_{2^*}^{(k)})$ are related by

$$\begin{aligned} J_1^{(k)} &= J_{1^*}^{(k)} \cos\varphi_k - J_{2^*}^{(k)} \sin\varphi_k, \\ J_2^{(k)} &= J_{1^*}^{(k)} \sin\varphi_k + J_{2^*}^{(k)} \cos\varphi_k, \end{aligned} \tag{7}$$

where the subscript $*$ denotes the quantities in the system $(x_{1^*}^{(k)}, x_{2^*}^{(k)})$ shown in Fig. 1.

It is well known that (Herrmann and Herrmann, 1981)

$$\begin{aligned} J_{1^*}^{(k)} &= \frac{\kappa + 1}{8\mu} \left[\left(K_{\mathrm{IR}}^{(k)}\right)^2 + \left(K_{\mathrm{IIR}}^{(k)}\right)^2 - \left(K_{\mathrm{IL}}^{(k)}\right)^2 - \left(K_{\mathrm{IIL}}^{(k)}\right)^2 \right], \\ J_{2^*}^{(k)} &= \frac{\kappa - 1}{4\mu} \left[K_{\mathrm{IL}}^{(k)} K_{\mathrm{IIL}}^{(k)} - K_{\mathrm{IR}}^{(k)} K_{\mathrm{IIR}}^{(k)} \right] + F_{2ak}^{*}, \end{aligned} \tag{8}$$

where F_{2ak}^{*} denotes the contribution of the traction-free surface of the kth microcrack to the second component of the J_k vector and

$$F_{2ak}^{*} = \int_{-a_k}^{a_k} (w^+ - w^-)\, \mathrm{d}x_{1^*}^{(k)}, \tag{9}$$

where w^+ and w^- denote the boundary values on the upper and lower surfaces of the kth microcrack, respectively, which could be calculated after the interaction problem shown in Fig. 1 is solved (Zhao and Chen (1997a,b) or Appendix B).

Obviously, the values of the J_k vector are not dependent on the location of the coordinate system or the origin selection of the system. Therefore, in the global system (x_1, x_2), the values of J_k could be evaluated by using a simple summation among $J_k^{(k)}$ derived in the local system $(x_1^{(k)}, x_2^{(k)})$

$$J_1 = \sum_{k=1}^{N} J_1^{(k)} = \sum_{k=1}^{N} (J_{1*}^{(k)} \cos\varphi_k - J_{2*}^{(k)} \sin\varphi_k), \tag{10}$$

$$J_2 = \sum_{k=1}^{N} J_2^{(k)} = \sum_{k=1}^{N} (J_{1*}^{(k)} \sin\varphi_k + J_{2*}^{(k)} \cos\varphi_k). \tag{11}$$

It is proved numerically by the present author and the co-worker (Chen and Hasebe, 1998) that the total contribution of the N microcracks to the J_1 and J_2 should vanish due to the remote uniform loading conditions with no other discontinuities outside the closed contour C (also see e.g., Herrmann and Herrmann, 1981 for one single crack, Zhao and Chen, 1997a for multiple subinterface cracks):

$$J_1 = \sum_{k=1}^{N} J_1^{(k)} = 0, \tag{12}$$

$$J_2 = \sum_{k=1}^{N} J_2^{(k)} = 0, \tag{13}$$

or

$$\sum_{k=1}^{N} (J_{1*}^{(k)} \cos\varphi_k - J_{2*}^{(k)} \sin\varphi_k) = 0, \tag{14}$$

$$\sum_{k=1}^{N} (J_{1*}^{(k)} \sin\varphi_k + J_{2*}^{(k)} \cos\varphi_k) = 0. \tag{15}$$

Indeed, Eqs. (12) and (13) or Eqs. (14) and (15) provide new conservation laws of the J_k vector in multiple crack interaction problems. The major assumption is that there are no other discontinuities outside of the closed contour C chosen to calculate the J_1 and the J_2 integrals.

As so many people, to whom the present author discussed, suspect these laws (12) and (13), it seems necessary to present a detailed proof to clarify the validity of the laws, which is based on such an important concept as how to use the remote uniform stress–strain field correctly. See Fig. 1, where N cracks are formed in a 2D-brittle solid. Introduce a closed but sufficient large contour $\Gamma_\infty = \Gamma_1 + \Gamma_L + \Gamma_2 + \Gamma_R$ surrounding all the cracks and introduce a smaller closed contour C_k only surrounding the kth crack completely (Fig. 1). According to the path-independent nature of the J_k integral vector in the present situation, it follows that

$$J_1^{\infty} = \sum_{k=1}^{N} J_1^{(k)}, \tag{16a}$$

$$J_2^{\infty} = \sum_{k=1}^{N} J_2^{(k)}, \tag{16b}$$

where the left terms are calculated over Γ_∞, while the right terms are calculated over C_k with $k = 1, 2, \ldots, N$, respectively. Since every term in the summation on the right side of Eq. (16a) or (16b), generally speaking, is not equal to zero due to strong interaction among the cracks, it is not clear whether the summation of N terms on the right side vanishes or not, as many researchers suspected.

Here, in order to account the remote uniform stress–strain field as well as the remote displacement field, the closed rectangular contour Γ_∞ has been chosen as large as possible. It should be mentioned that the reason to introduce the rectangular closed contour rather than an arbitrary one is to significantly simplify mathematical manipulations given below. If it is not so, e.g., an arbitrary smooth closed contour C_0 (also as large as possible, see Fig. 1) is chosen instead of the rectangular one, the results calculated over Γ_∞ and over C_0 will be the same due to the path-independent nature of the vector. It should be mentioned also that the second component of the vector is path independent too in the present situation because the closed contour is always chosen to be either surrounding all the cracks or surrounding a typical crack completely (Herrmann and Herrmann, 1981). Although the traction-free surfaces of each crack leads to some nonzero contribution to the second component, the path-independent nature has not been altered when the closed contour is chosen in the present way (Herrmann and Herrmann, 1981).

Since $\mathrm{d}y = \mathrm{d}x_2 = 0$ in Γ_1 and Γ_2, and $\mathrm{d}x = \mathrm{d}x_1 = 0$ in Γ_L and Γ_R, the left-hand side of Eq. (16a) becomes

$$J_1^\infty = \int_{\Gamma 1+\Gamma 2} (-\sigma_{i2}^\infty n_2 \partial u_i/\partial x)\,\mathrm{d}s + \int_{\Gamma \mathrm{L}+\Gamma \mathrm{R}} (W\,\mathrm{d}y) + \int_{\Gamma \mathrm{L}+\Gamma \mathrm{R}} (-\sigma_{i1} n_1 \partial u_i/\partial x)\,\mathrm{d}s. \tag{16c}$$

Noting that $n_2 = 1$ in Γ_1 and $n2 = -1$ in Γ_2 and $n_1 = 1$ in Γ_R and $n_1 = -1$ in Γ_L, it follows that

$$J_1^\infty = \sigma_{i2}^\infty \left[\int_a^b u_{i,1}\,\mathrm{d}x - \int_d^c u_{i,1}\,\mathrm{d}x \right] + \left[\frac{1}{2} \sigma_{ij}^\infty \varepsilon_{ij}^\infty \left(\int_b^c \mathrm{d}y - \int_a^d \mathrm{d}y \right) \right] + \sigma_{i1}^\infty \left[\int_b^c u_{i,1}\mathrm{d}y - \int_a^d u_{i,1}\mathrm{d}y \right]. \tag{16d}$$

Obviously, the second term in Eq. (16d) vanishes because $[\int_b^c \mathrm{d}y - \int_a^d \mathrm{d}y] = 0$, while the first term depends on the value $[\int_a^b u_{i,1}\,\mathrm{d}x - \int_d^c u_{i,1}\,\mathrm{d}x]$ and the third term depends on the value $[\int_b^c u_{i,1}\,\mathrm{d}y - \int_a^d u_{i,1}\,\mathrm{d}y]$. It should be noted that $u_{i,1}$ in the integral is defined at infinity since the rectangular closed contour has been chosen as large as possible. It could be recognized that the remote displacement field should be linear with respect to both x- and y-axes so that the uniform strain field at infinity could be deduced. In fact, the remote uniform strain field does not depend on the configuration of the N cracks (whatever the array of the cracks is symmetrical with respect to $x = 0$ (or $y = 0$) or not). Therefore, the asymptotic values at infinity $u_{i,1}$ should be constants and the value of $[\int_a^b u_{i,1}\,\mathrm{d}x - \int_d^c u_{i,1}\,\mathrm{d}x]$ should vanish.

This conclusion will become much clearer when dividing the original problem shown in Fig. 1 into two subproblems. First of them involves no cracks and the remote displacement field should be linear with respect to both x- and y-axes, while the second involves N cracks with self-balance tractions acting on both surfaces of each crack. The detailed configuration of such a subdivision is well known, which is no longer discussed here. Nevertheless, the second subproblem yields zero remote displacement field at infinity, because the stresses as well as the strains induced from the self-balance tractions have an asymptotic nature with the order of r^{-2} (for large values of $r = \sqrt{x^2 + y^2}$). Thus, the remote displacement field is only dominated by the first subproblem, which could certainly lead to the results, $u_{i,1}(\text{on } \Gamma_1) = u_{i,1}(\text{on } \Gamma_2)$. Indeed, this means that the first term or the third term in Eq. (16d) vanishes too, no matter how many cracks are formed in the finite region shown in Fig. 1 and whatever the array of the cracks is symmetrical with respect to $x = 0$ (or $y = 0$) or not. Therefore, Eq. (12) has been proved to be a new conservation law of the J_1 $(= J)$ integral. In fact, its validity has already been confirmed numerically by Chen and Hasebe (1998). Similarly, a straightforward manipulation could be given for Eq. (13), which represents a new conservation law of the J_2 integral and is no longer repeated here.

Obviously, such a vanishing nature of the vector does not depend on the shapes of the discontinuities enclosed by the closed contour C (Fig. 1). Therefore, other discontinuities such as curve cracks, crack bifurcations, crack growth, voids with any shapes, and inclusions, as enclosed by the contour C, will lead to the same conclusions formulated by Eqs. (12) and (13). Of course, the detailed expresses of the contributions induced from different shapes of discontinuities to the vector should be quite different, but the total summation of the contributions should be zero. This topic is beyond the scope of the present investigation.

It will be seen below that Eqs. (12) and (13) or Eqs. (14) and (15) provide two basic identities in the present situation. They do lead to the independence of the value of M from the origin location of global coordinate systems.

Turing back to the M-integral analysis, it follows that

$$\begin{aligned} M &= \sum_{k=1}^{N} M^{(k)}(x_1, x_2) = \sum_{k=1}^{N} \left\{ \oint_{C_k} (w x_i n_i - T_l u_{l,i} x_i)\,\mathrm{d}s \right\}, \\ &= \sum_{k=1}^{N} \left\{ \oint_{C_k} \left[w\left(x_i^{(k)} + \xi_i^{(k)}\right) n_i - T_l u_{l,i} \left(x_i^{(k)} + \xi_i^{(k)}\right) \right] \mathrm{d}s \right\}, \\ &= \sum_{k=1}^{N} \left\{ M^{(k)}\left(x_1^{(k)}, x_2^{(k)}\right) \right\} + \sum_{k=1}^{N} \left\{ \xi_1^{(k)} J_1^{(k)} + \xi_2^{(k)} J_2^{(k)} \right\} = M_{\mathrm{N}} + M_{\mathrm{A}}, \end{aligned} \tag{16e}$$

where $M^{(k)}(x_1^{(k)}, x_2^{(k)})$ is defined by Eq. (3) in the local system $(x_1^{(k)}, x_2^{(k)})$ and its value is calculated by Eq. (4), while $M^{(k)}(x_1, x_2)$ is considered in the global system (x_1, x_2). The subscripts N and A denote the net part and the additional part of the M integral.

Intuitively, the first term in Eq. (16e), denoted by M_{N} and called as the net contribution of the N cracks, does not influence the dependence of the M integral on the origin selection. Special attention should only be devoted to the second term in the right-hand side of Eq. (16e), since it involves the global coordinates of each crack center as well as the components of the J_k vector contributed from the formation of each crack. This term, denoted by M_{A}, has been called as the additional contribution of the N microcracks to the M integral in the global system. Due to its existence, a considerable confusion arises. Many people believe that the value of the M integral is dependent on the origin selection of the global coordinate system (x_1, x_2) since different selections of the origin of the global coordinates lead to difference values of $\xi_1^{(k)}$ and $\xi_2^{(k)}$. This is the major reason why the present author should perform so lengthy and so cumbersome manipulations to clarify the confusion mentioned above. Only after doing so, could the role M integral plays be discussed.

Perform another coordinate transformation from (x_1, x_2) to (x_{01}, x_{02}). The later is parallel to the former but originated at another point O_0 as shown in Fig. 1. The simple relations between the two coordinate systems are:

$$\begin{aligned} x_1 &= x_{01} - \eta_1, \\ x_2 &= x_{02} - \eta_2. \end{aligned} \tag{17}$$

The value of the M integral in the new global coordinate system (x_{01}, x_{02}) denoted by M_0 should be given as

$$\begin{aligned} M_0 &= \oint_{C} (w x_{0i} n_i - T_l u_{l,i} x_{0i})\,\mathrm{d}s, \\ &= \sum_{k=1}^{N} \left\{ M^{(k)}\left(x_1^{(k)}, x_2^{(k)}\right) + \left(\xi_1^{(k)} + \eta_1\right) J_1^{(k)} + \left(\xi_2^{(k)} + \eta_2\right) J_2^{(k)} \right\}, \\ &= M + \eta_1 \sum_{k=1}^{N} J_1^{(k)} + \eta_2 \sum_{k=1}^{N} J_2^{(k)}. \end{aligned} \tag{18}$$

Obviously, the last two terms in the right side of Eq. (18) vanish due to the conservation laws of the J_k vector formulated by Eqs. (12) and (13) for multi-cracks interaction. It is then concluded that

$$M_0 = M. \tag{19}$$

In other words, the value of the M integral does not depend on the origin selection of the global coordinates, either on the movement of the origin, or on the coordinate rotation. The later conclusion could be seen in the work done by Park and Earmme (1986) (also see Appendix A). It should be emphasized that Eq. (19) is directly deduced from the new conservation laws (12) and (13) with no regards to the details of micro-defects. This means that other defects such as curve cracks, growing cracks, voids with any shapes, branched crack, and crack bifurcation during loading history do not alter the independence of the M integral from the origin selection if all of them are enclosed by the closed contour C in Fig. 1. Nevertheless, different kinds of defects may contribute quite different values of the M integral and the growing microcracks enclosed by C may increase the values of the M integral. This topic is beyond the scope of the present investigation, which will be discussed in quite detail in the author' separated paper.

3. Numerical examples

Two numerical examples are given in this section to provide necessary evidences of the role the M integral plays in microcrack damage problems. The first is concerned with four regularly distributed microcracks, while the second with 20 randomly distributed microcracks.

3.1. Regularly distributed microcracks

Consider four strongly interacting microcracks regularly distributed in a plane elastic body as shown in Fig. 2. The body is loaded by the remote tensile stress σ_0 inclined by an angle ψ with respect to the x_{02}-axis. Here, $2d$ and $2L$ refer to the distance between microcrack centers, $2a$ refers to the length of each microcrack, ϕ denotes the oriented angle of each crack with respect to the x_{01}-axis. Two coordinate systems, i.e.,

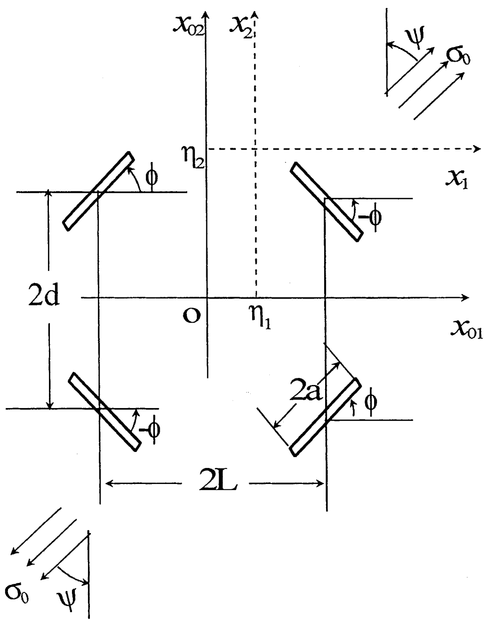

Fig. 2. Four regularly arranged microcracks under the remote inclined tensile loads.

(x_{01}, x_{02}) and (x_1, x_2) are introduced, which are related by Eq. (17), taking $\eta_1/a = 0.8$, $\eta_2/a = 2.3$, $L/a = 1.2$, and $d/a = 1.2$. The four microcracks are assumed to be fully open. This assumption could be ensured when the values of the Mode I stress-intensity factor at every tip are positive. It should be mentioned here that the purpose to present this example is not to study a real microcrack damage cell, rather, the purpose is to confirm the above derived independence of the M integral and to show the inherent relation between the value of the M integral and the reduction of the effective moduli arising from the formation of the four cracks.

Calculated values of the M integral are normalized by

$$M_{\mathrm{R}} = \frac{\kappa + 1}{8\mu} \cdot \pi \cdot (\sigma_0 a_0)^2, \tag{20a}$$

where a_0 refers to the average half length of all the microcracks. In this example, $a_0 = a$. The subscript R refers to the remote loading level.

In order to ensure the assumption mentioned in Section 1 that all the cracks are stationary, non-growing, the remote loading level is chosen in such a way that the numerical values of SIFs at every tip do not become critical. The critical condition is formulated below

$$(K_{\mathrm{I}}/K_{\mathrm{Ic}})^2 + (K_{\mathrm{II}}/K_{\mathrm{IIc}})^2 = 1, \tag{20b}$$

where the subscript c denotes the critical parameters of a certain brittle material. For example, for a certain kind of Al_2O_3 ceramic, $K_{\mathrm{Ic}} = 0.259$ MN/m$^{3/2}$ and $K_{\mathrm{IIc}} = 0.518$ MN/m$^{3/2}$. The assumption of non-growing cracks is actually ensured when taking the half crack length a_0 to be 0.001 m and the remote loading σ_O less than 2.737 MN/m^2.

In fact, the conclusion derived in the above section could be derived in a quite different way based on the energy-balance concept. As discussed by Herrmann and Herrmann (1981) for single crack problems, the change of the total potential energy (CTPE) due to the formation of a crack in a plane elastic body could be expressed by

$$U = \int_{-a}^{a} \sigma_{i2}^{\infty} \Delta u_i(x) \mathrm{d}x \quad (i = 1, 2), \tag{21}$$

where U refers to CTPE, σ_{i2}^{∞} refers to the remote stress, $\Delta u_i(x)$ is the displacement jump along the crack surfaces.

By using the relations among σ_{i2}^{∞}, $\Delta u_i(x)$ and M for a single crack in a uniform remote stress field (Herrmann and Herrmann, 1981), it is easy to prove that

$$M = 2U. \tag{22}$$

However, as mentioned earlier, the M integral for multi-interacting cracks could not be given by using a simple summation (Eqs. (5) and (16e)). Therefore, it is not clear whether or not Eq. (22) is valid for multi-cracks situations since the displacement jump along the surfaces of each crack is not only induced from the remote loading, but also disturbed by the neighboring and interacting microcracks. In these situations, the CTPE should be

$$U = \sum_{k=1}^{N} \int_{-a}^{a} \sigma_{i2}^{\infty}(k) \Delta u_i^{(k)} \left(x_{1^*}^{(k)} \right) \mathrm{d}x_{1^*}^{(k)}, \tag{23}$$

where the local coordinate $x_{1^*}^{(k)}$ is chosen for the kth crack only, $\sigma_{i2}^{\infty}(k)$ refers to the released stresses when the kth crack is formed, N is the number of microcracks, and $\Delta u_i^{(k)}(x_{1^*}^{(k)})$ is the displacement jump along the kth crack surfaces induced from the remote loading as well as the interaction with other cracks. It is proved numerically (Table 1) and mathematically (Appendix C) that the formulation (22) is still valid. This means that, from the physical point of view, the M integral is identically equal to twice of CTPE such that a

Table 1
Normalized values of the M integral and the CTPE due to the formation of the four microcracks shown in Fig. 2 taking $\phi = 15°$

ψ (°)	M_N	M_A	M	M_0	$2U$	M_T
0	3.5686	−0.0828	3.4858	3.4858	3.4858	3.7321
6	3.5442	−0.0875	3.4567	3.4567	3.4567	3.6942
10	3.5006	−0.0955	3.4051	3.4051	3.4051	3.6276
16	3.3931	−0.1128	3.2803	3.2803	3.2803	3.4689
30	2.9405	−0.1571	2.7834	2.7834	2.7834	2.8660
46	2.1080	−0.1596	1.9484	1.9484	1.9484	1.9396
60	1.2493	−0.0972	1.1521	1.1521	1.1521	1.1340
76	0.4479	−0.0006	0.4473	0.4473	0.4473	0.4707
90	0.1860	0.0371	0.2231	0.2231	0.2231	0.2680

definite value of the M integral exists for multi-strongly interacting cracks. Thus, the value does, of course, not depend on the selection of the global coordinates, either on the origin location, or on the coordinate rotation. Moreover, this leads to a simple relation between the M integral and the L integral since Herrmann and Herrmann (1981) showed that the L integral represents the rotation energy release rate:

$$L = -\tfrac{1}{2}\partial M/\partial\psi. \qquad (24)$$

This relation has never been reported before.

Table 1 shows that the values of the M integral calculated respectively in the two coordinate systems (x_{01}, x_{02}) and (x_1, x_2) (denoted by M_0 and M, respectively) coincide well with each other, which support the conclusion (19) given in Section 2. Moreover, it is seen that Eq. (22) is valid in the four microcrack interacting case shown in Fig. 2. This reveals that the M integral for a microcracking solid represents the change of the total potential energy due to the formation of the microcracks. It could be imagined that the M integral for an evolutionary damage represents the progressive energy release due to damage growth (i.e., microcrack growth, microcrack coalescence, new microcrack nucleation) (Chaboche, 1988a,b; Jun and Chen, 1994a,b). This topic will no longer be discussed here for shortening the length of this paper.

Of great interest are the variable tendencies of the normalized values of the M integral against the loading angle as shown in Fig. 3(a)–(g) taking the oriented angle ϕ to be 0°, 15°, 30°, 45°, 60°, 75°, and 90°, respectively. Here, M_N denotes the net contribution of stress intensity factors at the eight tips of the four cracks to the M integral, i.e., the first term in Eq. (16e). M_A denotes the additional contribution induced from the coordinates of the microcrack centers and the $J_k^{(k)}$ vector of each crack to the M integral, i.e., the second term in Eq. (16e). $M\ (= M_0)$ is the summation between M_N and M_A. M_T is specially introduced for comparisons, which is calculated under the assumption of "Taylor's Models" (Jun and Chen, 1994a), where the microcrack interactions are entirely neglected.

It is seen from detailed comparisons between Fig. 3(a) and (g), Fig. 3(b) and (f), and Fig. 3(c) and (e) that each pair shows mirror effect with respect to the line of $\psi = 45°$ when the inclined loading angle ψ increases from 0° to 90°. These results confirm the independence of the values of the M-integral from the rotation of the preferred coordinate system. A certain loading angle in Fig. 3(a)–(c), say $\psi = 16°$ is just corresponding to another certain loading angle in Fig. 3(g), (f) and (e) say, $\psi = 90 - 16 = 74°$ when the coordinate system rotates 90°.

Of the most interest is the implicit relation between the values of M and the effective elastic moduli for the microcracking solid. It is found that the maximum value of the M integral is just corresponding to the direction along which the largest reduction of the effective elastic moduli occurs. For example, $\varphi = 0°$ in Fig. 3(a)–(c) or $\varphi = 90°$ in Fig. 3(e)–(g). Quite contrary, the minimum value of the M integral is just corresponding to the direction along which the smallest reduction of the effective elastic moduli occurs. For example, $\varphi = 90°$ in Fig. 3(a)–(c) or $\varphi = 0°$ in Fig. 3(e)–(g). Indeed, there really exists an inherent relation

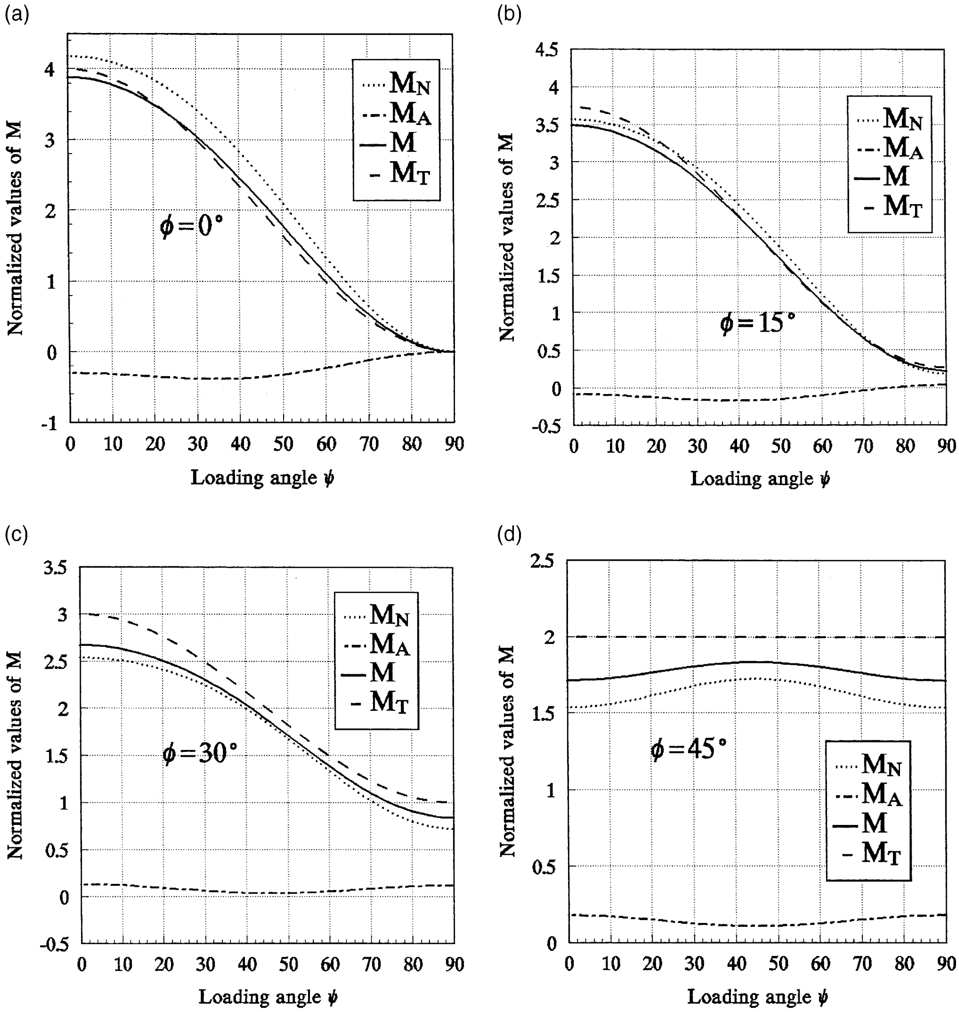

Fig. 3. (a)–(g). Normalized values of the M integral and the CTPE against the loading angle ψ taking different values of the microcrack oriented angle ϕ.

between the values of the M integral and the effective elastic moduli induced from microcracking along different tensile loading directions. The larger value of the M integral along a tensile loading direction is, the larger reduction of the effective elastic moduli along the direction is. On contrary, Fig. 3(d) shows that the values of the M integral are not sensitive to the tensile loading direction since the four microcrack configuration with $\varphi = 45°$ represents a nearly uniform reduction of the effective elastic moduli.

It is also seen from Table 1 and Fig. 3(a)–(g) that the divergence between the values of the M integral and those of M_T sometimes is remarkable due to neglecting the interacting effect among the four microcracks (where M_T denotes the values calculated by using Talor's model). It is concluded that the divergence between Talor's model and that of Jun and Chen (1994a,b) could be evaluated by using the M-integral analysis. In most cases from Fig. 3(a)–(g), Talor's model overestimates the values of the M integral so that overestimates the reduction of the effective elastic moduli for microcrack weakened brittle solids.

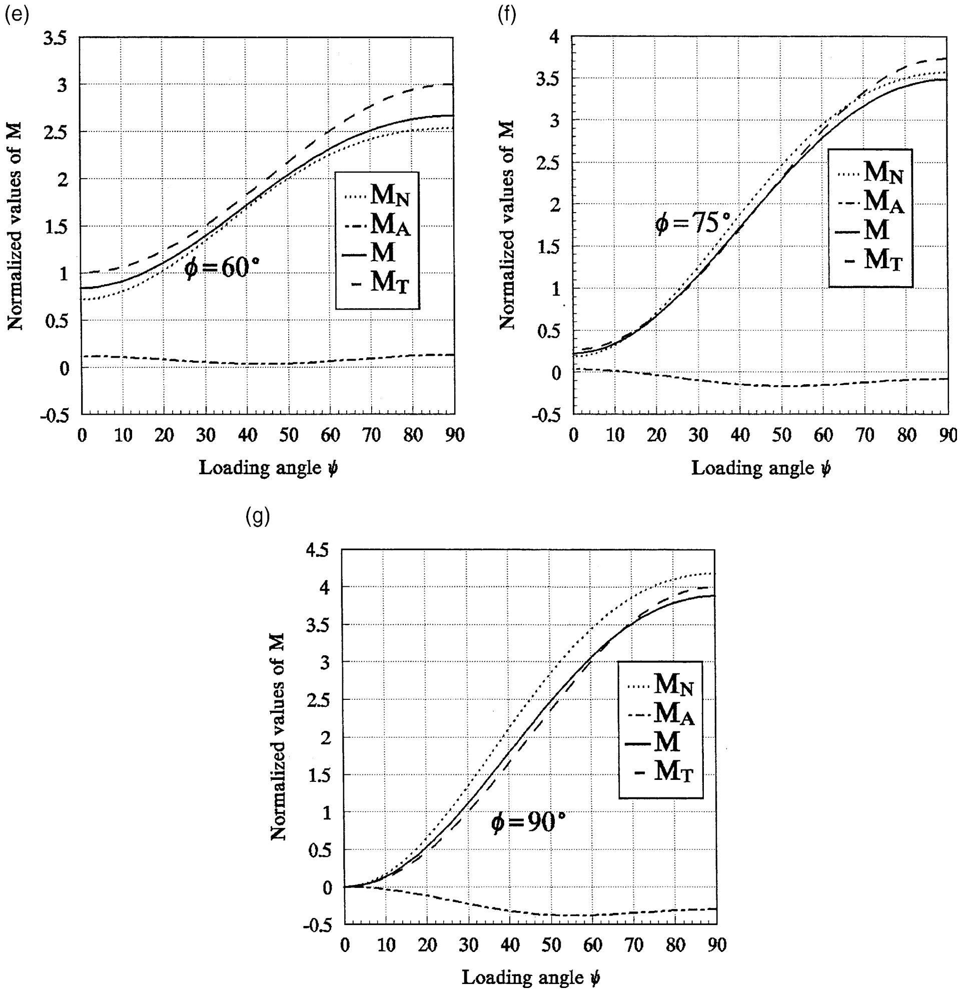

Fig. 3. (*continued*)

3.2. Randomly distributed microcracks

Consider 20 strongly interacting microcracks randomly distributed in a plane elastic body as shown in Fig. 4(a)–(d). Calculated values of the M integral are also normalized by Eq. (20a). Moreover, the remote tensile loading is chosen in such a way that the stationary conditions of all the cracks should be satisfied. This could be done by substituting the values of the SIFs at each tip of every crack one by one into Eq. (20b) to ensure the tensile loading be less than the critical value.

Fig. 5(a)–(d) show the variable tendencies of the M integral against the loading angle, ψ for the crack arrays shown in Fig. 4(a)–(d), respectively. It is seen that the values of the M integral are not sensitive to the tensile loading direction. This conclusion coincides well with the well-known fact (Jun and Chen, 1994a,b) that microcrack-weakened brittle solids show an isotropic nature, when microcracks are sufficiently randomly distributed although the effective elastic moduli are reduced significantly. It could be seen also that

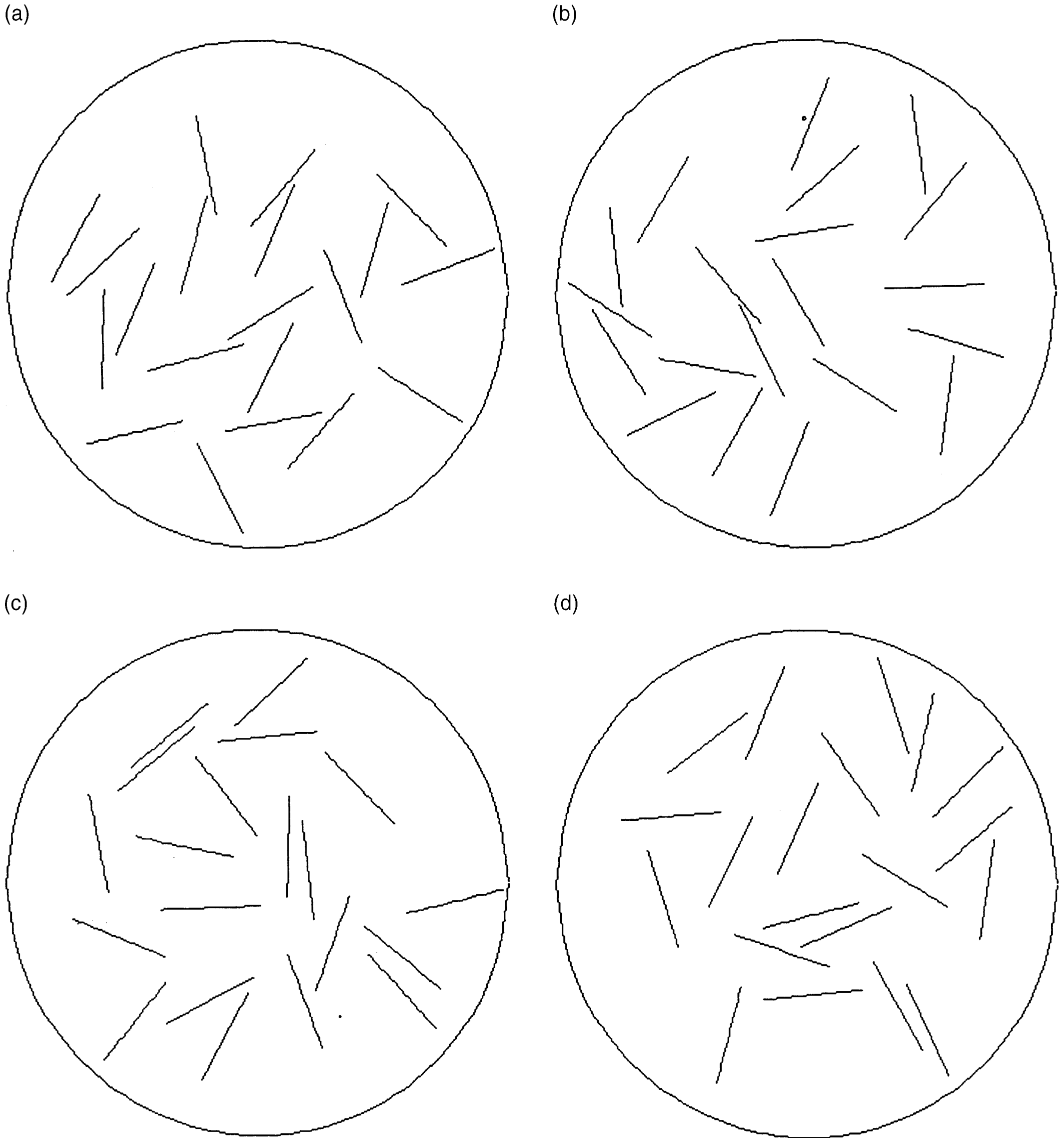

Fig. 4. (a)–(d). Four kinds of 20 microcracks distributed randomly in a circular region.

the additional part M_A always takes an in-neglected contribution to the M integral under strongly interacting situations.

Of course, a real microcrack-weakened brittle solid in practice with, e.g., more than hundreds of microcracks is quite difficult to be evaluated by using the traditional fracture concept due to the intractable large amount of the computation. For this reason, number of researchers focus their attention on the effective elastic moduli to account stiffness and stability of the solid with strongly interacting microcracks. For example, Jun and Chen (1994a,b) used the mixed fracture criteria in their effective elastic moduli methods. However, as pointed out by Kachanov (1993), what controls stiffness and what drives fracture in

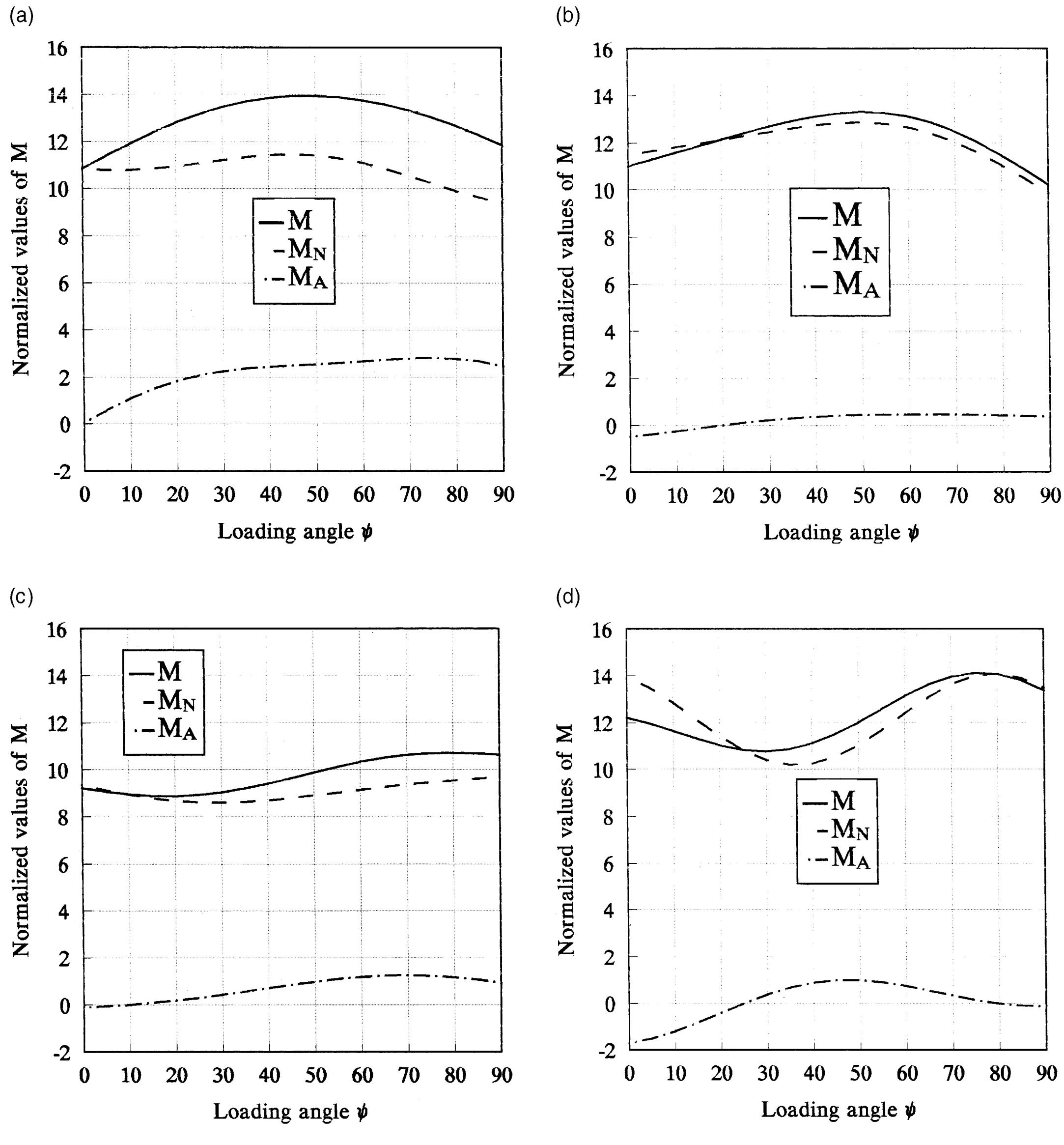

Fig. 5. (a)–(d). Normalized values of the M integral against the loading angle ψ for the four kinds of the 20 microcracks.

microcrack array settings is quite different. The first may shed little light on the second. From the discussions mentioned above, the present investigation does present a new way, from the phenomenological point of view, to describe the microcrack-weakened problems in brittle solids. Although the present investigation is an initial attempt, which has not been proved to be universal to include all aspects of such problems, it does provide an evidence and a possibility, that there exists an alternative evaluation for microcracking brittle solids based on the M-integral analysis. Therefore, researchers may perform more efforts on such a new description to study the nonlinear mechanical behaviors of microcracking brittle solids. For example, the nondestructive technique (King and Herrmann, 1981) customarily used for single crack problems could be adopted to account the damage levels of microcracking brittle solid in experimental practices.

4. Conclusions and remarks

From the above performed manipulations and numerical examinations, the following conclusions could be given:

(1) There exist new conservation laws of the J_k vector for an infinite 2D microcracking brittle solid. The laws reveal that the total contributions induced from the formation of multi-cracks in the solid to the two components of the vector vanish, providing that the closed contour chosen to calculate the vector encloses all the microcracks or there are no other discontinuities outside of the closed contour.

(2) The M integral calculated along a closed contour surrounding all microcracks in an infinite 2D microcracking solid is divided into two distinct parts. One of them (called the net part) is contributed by microcrack tip stress intensity factors as addressed by Freund (1978), and the other (called the additional part) is contributed by the microcrack center coordinates and the J_k vector defined for each microcrack. Although the additional part involves the microcrack center coordinates, the M integral does not depend on the selection of the global coordinate system, either on the origin selection or on the coordinate rotation. This conclusion is directly deduced from the conservation laws of the J_k vector.

(3) Although Kanninen and Popelar (1985) pointed out that there has been apparent little effort to apply the M integral in practical problems, the present investigation reveals that the M integral does play an important role in the description of damaged brittle solids due to microcracking. Moreover, it does provide an effective measure for the evaluation of damage level in some examples shown in this paper. In other words, the present investigation provides some important evidences that micro-structural statistical information is actually embedded in the formulation of a phenomenological parameter: the M integral.

(4) The dependence of the M integral values on the remote tensile loading direction reveals that the maximum value of the integral for a certain microcrack array coincides with the maximum loss direction in stiffness, while the minimum value coincides with the minimum loss direction. This means that, from the physical point of view, the M integral (whose value has apparent directional nature with respect to different loads) is inherently related with the effective elastic moduli for stationary microcracking solids. Of course, the detailed relation between the effective elastic moduli and the values of the M integral remains to be adequately investigated.

(5) The M integral is proved numerically and mathematically to be identically equal to twice of the decrease of the total potential energy due to the formation of the pre-existing microcracks. Thus, there should be a simple relation between the M integral and the L integral, i.e., Eq. (24), providing that the closed contour chosen to calculate the two integrals encloses all the microcracks or there are no other discontinuities outside the contour.

(6) Potential applications of the new description based on the M-integral analysis are remarkable. For example, the increasing values of the M integral during an evolutionary microcrack damage just represent the progressive energy release due to the damage growth, e.g., microcrack growth, microcrack coalescence, and new microcrack nucleation. This topic will inevitably require further investigation and will be discussed in the author's separate paper.

(7) The restriction of the present investigation is considerable since the conservation laws are only valid for an infinite brittle solid. This restriction could be removed by evaluating the contribution induced from the interface between two dissimilar materials to the J_k vector, from which new formulations of the conservation laws are derived instead of Eqs. (12) and (13). Only in this way, could the outside boundaries of a finite microcracking solid be considered as a special kind of interface between air and the solid or between a rigid body and the solid. This topic will be discussed in Part II of this series.

Acknowledgements

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Appendix A

Perform a coordinate transformation from (x_1, x_2) to (x_1^*, x_2^*) with the same origin, but with an oriented angle φ. The M integral becomes the M^* integral and the outside normal n_i is related by

$$\begin{aligned} n_1 &= n_1^* \cos\varphi - n_2^* \sin\varphi, \\ n_2 &= n_1^* \sin\varphi + n_2^* \cos\varphi. \end{aligned} \tag{A.1}$$

Thus,

$$x_i n_i = x_1 n_1 + x_2 n_2 = x_1^* n_1^* + x_2^* n_2^*, \tag{A.2}$$

$$T_l u_{l,i} x_i = T_l^* u_{l,i}^* x_i^*, \tag{A.3}$$

and

$$\begin{aligned} &\oint_C (W x_i n_i - T_l u_{l,i} x_i)\,\mathrm{d}s = \oint_C (W x_i^* n_i^* - T_l^* u_{l,i}^* x_i^*)\,\mathrm{d}s, \\ &\text{i.e.}\quad M = M^*, \end{aligned} \tag{A.4}$$

where the superscript * denotes the quantities in the coordinate system (x_1^*, x_2^*).

It is proved that the value of the M integral does not depend on the rotation of the coordinated system.

Appendix B

As discussed by Herrmann and Herrmann (1981), the traction-free surfaces of a crack have some contribution to the second component of the J_k vector. For the kth microcrack shown in Fig. 1, this contribution could be calculated in the local system $(x_{1^*}^{(k)}, x_{2^*}^{(k)})$

$$F_{2a_k}^{a_k} = \int_{-a_k}^{a_k} (W^+ - W^-)\,\mathrm{d}x_{1^*}^{(k)}, \tag{B.1}$$

where W^+ and W^- indicate the upper and lower boundary values along the crack surfaces of the strain energy, which could be given as

$$\begin{aligned} W^+ &= \frac{1-v^2}{2E}(\sigma_{11^*}^+(t))^2, \\ W^- &= \frac{1-v^2}{2E}(\sigma_{11^*}^-(t))^2, \end{aligned} \tag{B.2}$$

where t denotes $x_{1^*}^{(k)}$,$\sigma_{1^*}^+(t)$ and $\sigma_{1^*}^-(t)$ are the normal stresses parallel to the upper and lower crack surfaces, respectively.

A special numerical technique was introduced by Zhao and Chen (1997a,b), from which a deduced singular integral, i.e.,

$$I = \frac{1}{2\pi \mathrm{i}\sqrt{a_k^2 - t^2}} \int_{-a_k}^{a_k} \frac{P(s) - \mathrm{i}Q(s)}{t - s} \sqrt{a_k^2 - s^2}\, \mathrm{d}s \tag{B.3}$$

could be calculated. Here, $P(s)$ and $Q(s)$ are the so-called pseudo-tractions acting on both surfaces of the kth microcrack.

Eq. (B3) is normalized as

$$I = \frac{1}{2\pi \mathrm{i}\sqrt{1 - (t/a_k)^2}} \int_{-1}^{1} \frac{P(u) - \mathrm{i}Q(u)}{(t/a_k - u)} \sqrt{1 - u^2}\, \mathrm{d}u, \tag{B.4}$$

where

$$u = s/a_k. \tag{B.5}$$

Expanding $P(u)$−i$Q(u)$ into the second kind of the Chebyshev Polynomial $U_j(u)$ as follows:

$$P(u) - \mathrm{i}Q(u) = \sum_{j=1}^{M-1} B_j U_j \tag{B.6}$$

with

$$B_j = \frac{2}{\pi} \int_{-1}^{1} \sqrt{1 - u^2}(P(u) - \mathrm{i}Q(u)) U_j(u)\, \mathrm{d}u, \tag{B.7}$$

and substituting Eq. (B6) into Eq. (B4), the following numerical technique is obtained

$$\begin{aligned} I &= \frac{1}{2\pi \mathrm{i}\sqrt{1 - (t/a_k)^2}} \int_{-1}^{1} \frac{P(u) - \mathrm{i}Q(u)}{(t/a_k - u)} \sqrt{1 - u^2}\, \mathrm{d}u, \\ &= \frac{1}{2\mathrm{i}\sqrt{1 - (t/a_k)^2}} \sum_{j=0}^{M-1} B_j T_{j+1}(t/a_k), \end{aligned} \tag{B.8}$$

where $T_{j+1}(t/a_k)$ is the first kind of the Chebyshev polynomial.

Appendix C

The relation between the M integral and the change of the total potential energy is clarified below by taking the integral closed contour as one runs nicely along surfaces of each crack, s_k ($k = 1,2,\ldots,N$). Due to the traction-free condition on the surface, the expression of the M integral (1) is deduced to

$$M = \oint_s w x_i n_i\, \mathrm{d}l, \tag{C.1}$$

where $s = \sum_{k=1}^{N} s_k$ is the concourse of all closed contours along the surfaces of each crack.

According to the work of Budiansky and Rice (1973), a quasi-elastostatic boundary-value problem associated with the 2D solid contained within the surface $S + s$ is considered. Here, s refers to the traction-free surfaces and the external loading is imposed only by traction on S.

Without changing the boundary conditions on S, the continuously varying sequence of static solutions for the displacement $\mathbf{u}$ could be contemplated, which is generated as the plane specification of s is varied with a time-like parameter t. The potential energy of the system at any time is

$$\Pi = \int_{V(t)} w\,\mathrm{d}V - \int_{V(t)} f_i u_i\,\mathrm{d}V - \int_S X_i n_i\,\mathrm{d}s, \tag{C.2}$$

where $V(t)$ is the volume enclosed by $S + s(t)$, and, f_i, X_i, and u_i are the body force, the traction acting on S, and the displacement in the body, respectively. Thus,

$$\frac{\mathrm{d}\Pi}{\mathrm{d}t} = \dot{\Pi} = \int_{V(t)} \dot{w}\,\mathrm{d}V + \int_{V(t)} w\frac{\mathrm{d}x_i}{\mathrm{d}t}n_i\,\mathrm{d}s - \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{V(t)} f_i u_i\,\mathrm{d}V + \int_S X_i n_i\,\mathrm{d}S\right). \tag{C.3}$$

Assuming that $\mathrm{d}u_i/\mathrm{d}t$ is an admissible function in $V(t)$ and using the principle of the virtual work rate, the first term and the third term in Eq. (C3) cancel. Therefore,

$$\frac{\mathrm{d}\Pi}{\mathrm{d}t} = \int_{s(t)} w\frac{\mathrm{d}x_i}{\mathrm{d}t}n_i\,\mathrm{d}s. \tag{C.4}$$

Suppose that s is the boundary of the multi-cracks and let

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = x_i, \tag{C.5}$$

which directly leads to

$$\frac{\mathrm{d}\Pi}{\mathrm{d}t} = \int_s w x_i n_i\,\mathrm{d}s. \tag{C.6}$$

By comparing Eq. (C6) with Eq. (C1), it is clear that both equations are equivalent with each other. Moreover, the solution of Eq. (C5) could be formulated as

$$x_i = c_i \mathrm{e}^t. \tag{C.7}$$

It is noted that Eqs. (C6) and (C7) represent the self-similar expansion of the multi-cracks. This is the reason why Budiansky and Rice (1973) explained the M integral as the energy release rate associated with the self-similar expansion of a traction-free cavity. As known, the boundary s has a similar alteration form to x_i for plane problems, i.e., $s = s_0\mathrm{e}^t$ with s_0 as a reference boundary. Thus, w and u_i are independent of t in Eq. (C6). This directly leads to the following result

$$\Delta\Pi = \int_{-\infty}^{t} \frac{\mathrm{d}\Pi}{\mathrm{d}t}\mathrm{d}t = \int_{-\infty}^{t}\left(\mathrm{e}^{2t}\int_{s0} w c_i n_i\,\mathrm{d}s_0\right)\mathrm{d}t = \frac{1}{2}\left(\mathrm{e}^{2t}\int_{s0} w c_i n_i\,\mathrm{d}s_0\right) \quad \text{with } s_0 = s\mathrm{e}^{-t}. \tag{C.8}$$

It is obvious that Eq. (C8) with $s_0 = s\mathrm{e}^{-t}$ is equal to the CTPE of the body under consideration since the starting point and ending point of t represent the initial non-crack state and the present cracked state, respectively. By reconsidering Eq. (C6), it is concluded that

$$\mathrm{CTPE} = \Delta\Pi = \tfrac{1}{2}M. \tag{C.9}$$

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